ΓΕΝΙΚΕΣ ΑΣΚΗΣΕΙΣ Α

INT GEN-A1

Α. Χρησιμοποιώντας την αντικατάσταση που δίνεται, ή με άλλο τρόπο, να βρείτε τα ολοκληρώματα :

(Α1) $\int \frac{\sqrt{x}}{\sqrt{x}+3}\,dx$, $u=\sqrt{x}$

$$\frac{du}{dx}=\frac{1}{2\sqrt{x}} \Rightarrow dx=2\sqrt{x}\,du$$

Άρα

$$\int \frac{\sqrt{x}}{\sqrt{x}+3}\,dx=\int \frac{\sqrt{x}}{\sqrt{x}+3}\,2\sqrt{x}\,du$$

$$=\int \frac{u\cdot 2\cdot u}{u+3}\,du=\int \frac{2u^2}{u+3}\,du$$

$$=\int \left(2u-6+\frac{18}{u+3}\right)du$$

$$=\int 2u\,du-\int 6\,du+\int \frac{18}{u+3}\,du$$

$$=\frac{2u^2}{2}-6u+18\ln|u+3|+c$$

$$=(\sqrt{x})^2-6\sqrt{x}+18\ln|\sqrt{x}+3|+c$$

$$=x-6\sqrt{x}+18\ln|\sqrt{x}+3|+c$$

Εκτελούμε τη διαίρεση

$$\begin{array}{r|l} 2u^2 & u+3 \\ \cline{2-2} -2u^2-6u & 2u-6 \\ \hline -6u & \\ +6u+18 & \\ \hline 18 & \end{array}$$

Επομένως

$$\frac{2u^2}{u+3}=2u-6+\frac{18}{u+3}$$

INTGEN-A2

(A2) $\int \frac{x^5}{x^2+2}\,dx$, $x^2+2=u$

$du = 2x\,dx \Rightarrow dx = \frac{du}{2x}$

Επίσης $x^2 = u-2$

Άρα

$$\int \frac{x^5}{x^2+2}\,dx = \int \frac{x^5 \cdot \frac{du}{2x}}{x^2+2} = \int \frac{x^4\,du}{2u} =$$

$$= \int \frac{(x^2)^2\,du}{2u} = \int \frac{(u-2)^2}{2u}\,du = \int \frac{u^2-4u+4}{2u}\,du$$

$$= \int \left(\frac{u}{2} - 2 + \frac{2}{u}\right) du = \int \frac{u}{2}du - \int 2du + \int \frac{2}{u}\,du$$

$$= \frac{u^2}{4} - 2u + 2\ln|u| + c$$

$$= \frac{(x^2+2)^2}{4} - 2(x^2+2) + 2\ln(x^2+2) + c_1$$

$$= \frac{x^4+4x^2+4}{4} - 2x^2 - 4 + 2\ln(x^2+2) + c_1$$

$$= \frac{x^4}{4} + x^2 + 1 - 2x^2 - 4 + 2\ln(x^2+2) + c_1$$

$$= \frac{x^4}{4} - x^2 + 2\ln(x^2+2) + c$$

όπου $c = 1 - 4 + c_1$

(A3) $\int x^3 e^{x^2} dx$, $x^2 = u$

$$\frac{du}{dx} = 2x \Rightarrow dx = \frac{du}{2x}$$

Άρα

$$\int x^3 e^{x^2} dx = \int x^3 e^{x^2} \frac{du}{2x} = \frac{1}{2}\int x^2 e^{x^2} du$$

$$= \frac{1}{2}\int u e^u du = \frac{1}{2}\int u\, d(e^u)$$

$$= \frac{1}{2}\left[u \cdot e^u - \int e^u du\right] = \frac{1}{2}\left(ue^u - e^u\right) + c$$

$$= \frac{1}{2}x^2 e^{x^2} - \frac{1}{2}e^{x^2} + c$$

(A4) $\int \frac{x^3}{\sqrt{x^2+1}} dx$, $u = \sqrt{x^2+1} \Rightarrow du = \frac{2x}{2\sqrt{x^2+1}} dx$

$$\Rightarrow dx = \frac{\sqrt{x^2+1}}{x} du$$

Επίσης $u^2 = x^2+1$

$\Rightarrow x^2 = u^2 - 1$

Άρα $\int \frac{x^3}{\sqrt{x^2+1}} dx = \int \frac{x^3}{\sqrt{x^2+1}} \cdot \frac{\sqrt{x^2+1}}{x} du =$

$$= \int x^2 du = \int (u^2-1) du = \int u^2 du - \int du$$

$$= \frac{u^3}{3} - u + c = \frac{\left(\sqrt{x^2+1}\right)^3}{3} - \sqrt{x^2+1} + c$$

(A5) $\displaystyle\int \frac{x^2-2x+3}{(x-2)\sqrt{x-2}}\,dx \quad , \quad u=\sqrt{x-2}$

$$\Rightarrow \frac{du}{dx} = \frac{1}{2\sqrt{x-2}}$$

$$\Rightarrow dx = 2\sqrt{x-2}\,du$$

Επίσης $u^2 = x-2 \Rightarrow x = u^2+2$

$$\Rightarrow x^2 = u^4+4u^2+4$$

Άρα:

$$\int \frac{x^2-2x+3}{(x-2)\sqrt{x-2}}\,dx = \int \frac{u^4+4u^2+4-2(u^2+2)+3}{u\sqrt{x-2}} \cdot 2\sqrt{x-2}\,du$$

$$= \int \frac{u^4+4u^2+7-2u^2-4}{u} \cdot 2\,du$$

$$= 2\int \frac{u^4+2u^2+3}{u^2}\,du$$

$$= 2\left[\int u^2\,du + 2\int du + 3\int u^{-2}\,du\right]$$

$$= 2\left(\frac{u^3}{3} + 2u + 3\frac{u^{-1}}{-1}\right) + c$$

$$= \frac{2}{3}\left(\sqrt{x-2}\right)^3 + 4\sqrt{x-2} - 6\cdot\frac{1}{\sqrt{x-2}} + c$$

(A6) $\int \frac{ημ2x}{\sqrt{1+ημx}}dx$, $u=\sqrt{1+ημx}$

$$\frac{du}{dx}=\frac{συνx}{2\sqrt{1+ημx}} \Rightarrow dx=\frac{2\sqrt{1+ημx}}{συνx}du$$

Επίσης $u^2=1+ημx \Rightarrow ημx=u^2-1$

Άρα:

$$\int \frac{ημ2x}{\sqrt{1+ημx}}dx=\int \frac{2ημx\,συνx}{\sqrt{1+ημx}}\cdot\frac{2\sqrt{1+ημx}}{συνx}du=$$

$$=\int 4\,ημx\,du=2\int(u^2-1)du$$

$$=4\left[\frac{u^3}{3}-u\right]+c=\frac{4}{3}\left(\sqrt{1+ημx}\right)^3-4\sqrt{1+ημx}+c$$

(A7) $\int \frac{dx}{\sqrt{x+5}-1}$, $u=\sqrt{x+5}-1 \Rightarrow \sqrt{x+5}=u+1$

$$\Rightarrow \frac{du}{dx}=\frac{1}{2\sqrt{x+5}} \Rightarrow dx=2\sqrt{x+5}\,du$$

Επίσης $\sqrt{x+5}=u+1$

Άρα: $\int \frac{dx}{\sqrt{x+5}-1}=\int \frac{2\sqrt{x+5}\,du}{u}=\int \frac{2(u+1)}{u}du=$

$$=2\int\left(1+\frac{1}{u}\right)du=2\left[u+\ln|u|\right]+c_1$$

$$=2u+2\ln|u|+c_1=2\left(\sqrt{x+5}-1\right)+2\ln\left|\sqrt{x+5}-1\right|+c_1$$

$$=2\sqrt{x+5}-2+2\ln\left|\sqrt{x+5}-1\right|+c_1$$

$$=2\sqrt{x+5}+2\ln\left|\sqrt{x+5}-1\right|+c$$ όπου $c=c_1-2$

(A8) $\int \sqrt{\dfrac{x}{1-x}}\,dx$, $x = ημ^2θ$, $0 < θ < \dfrac{π}{2}$

$dx = 2ημθ \cdot συνθ \, dθ$

Άρα :

$$\int \sqrt{\frac{x}{1-x}}\,dx = \int \sqrt{\frac{ημ^2θ}{1-ημ^2θ}} \cdot 2ημθ\,συνθ\,dθ =$$

$$= \int \sqrt{\frac{ημ^2θ}{συν^2θ}}\, 2 \cdot ημθ\,συνθ\,dθ$$

$$= \int \frac{|ημθ|}{|συνθ|} \cdot 2\,ημθ\,συνθ\,dθ$$

$$= \int \frac{ημθ}{συνθ}\, 2 \cdot ημθ\,συνθ\,dθ$$

διότι $ημθ > 0$ και $συνθ > 0$ για $0 < θ < \dfrac{π}{2}$

$$= \int 2ημ^2θ\,dθ$$

$$= \int 2 \cdot \frac{1-συν2θ}{2}\,dθ = \int 1\,dθ - \int συν2θ\,dθ =$$

$$= θ - \frac{1}{2}ημ2θ + c$$

$$= θ - \frac{1}{2} \cdot 2ημθ\,συνθ + c$$

$$= τοξημ\sqrt{x} - \sqrt{x} \cdot \sqrt{1-x} + c$$

διότι $ημθ = \sqrt{x}$

$θ = τοξημ\sqrt{x}$

και $συνθ = \sqrt{1-ημ^2θ}$

$= \sqrt{1-x}$

(A9) $\int \frac{dx}{x^2\sqrt{9-x^2}}$, $x=3\eta\mu\theta$, $0<\theta<\frac{\pi}{2}$

$dx=3\sigma\upsilon\nu\theta\, d\theta$

Άρα

$$\int \frac{dx}{x^2\sqrt{9-x^2}} = \int \frac{3\sigma\upsilon\nu\theta\, d\theta}{(3\eta\mu\theta)^2\sqrt{9-(3\eta\mu\theta)^2}} =$$

$$= \int \frac{3\sigma\upsilon\nu\theta\, d\theta}{9\eta\mu^2\theta\sqrt{9-9\eta\mu^2\theta}}$$

$$= \int \frac{\sigma\upsilon\nu\theta\, d\theta}{3\eta\mu^2\theta\cdot 3\sqrt{1-\eta\mu^2\theta}} = \int \frac{\sigma\upsilon\nu\theta\, d\theta}{3\eta\mu^2\theta\cdot 3\sqrt{\sigma\upsilon\nu^2\theta}}$$

$$= \int \frac{\sigma\upsilon\nu\theta\, d\theta}{3\eta\mu^2\theta\cdot 3\cdot|\sigma\upsilon\nu\theta|}$$

$$= \int \frac{\sigma\upsilon\nu\theta\, d\theta}{9\eta\mu^2\theta\cdot\sigma\upsilon\nu\theta}$$ διότι $\sigma\upsilon\nu\theta>0$ για $0<\theta<\frac{\pi}{2}$

$$= \frac{1}{9}\int \frac{d\theta}{\eta\mu^2\theta} = \frac{1}{9}\int \sigma\tau\varepsilon\mu^2\theta\, d\theta$$

$$= -\frac{1}{9}\sigma\varphi\theta + c = -\frac{1}{9}\frac{\sigma\upsilon\nu\theta}{\eta\mu\theta} + c = -\frac{\sqrt{1-\eta\mu^2\theta}}{9\eta\mu\theta} + c$$

$$= -\frac{\sqrt{1-(\frac{x}{3})^2}}{9\cdot\frac{x}{3}} + c = -\frac{\sqrt{1-\frac{x^2}{9}}}{3x} + c$$

$$= -\frac{\sqrt{\frac{9-x^2}{9}}}{3x} + c = -\frac{\sqrt{9-x^2}}{9x} + c$$

διότι $\eta\mu\theta = \frac{x}{3}$

INTGEN-A10

(A10) $\int \frac{x^2}{\sqrt{(1-x^2)^3}}\,dx$, $x = ημθ$, $0 < θ < \frac{\pi}{2}$

$dx = συνθ\,dθ$

$$\int \frac{x^2}{\sqrt{(1-x^2)^3}}\,dx = \int \frac{ημ^2θ}{\sqrt{(1-ημ^2θ)^3}}\,συνθ\,dθ$$

$$= \int \frac{ημ^2θ\,συνθ\,dθ}{\sqrt{(συν^2θ)^3}} = \int \frac{ημ^2θ\,συνθ\,dθ}{συν^3θ}$$

$$= \int \frac{ημ^2θ}{συν^2θ}\,dθ = \int εφ^2θ\,dθ = \int (τεμ^2θ - 1)\,dθ$$

$$= \int τεμ^2θ\,dθ - \int dθ$$

$$= εφθ - θ + c$$

$$= \frac{ημθ}{συνθ} - θ + c = \frac{x}{\sqrt{1-x^2}} - τοξημx + c$$

διότι $θ = τοξημx$

και $συνθ = \sqrt{1-ημ^2θ}$

$= \sqrt{1-x^2}$

(A11) $\displaystyle\int \frac{1}{x^2(1+x^2)}\,dx \quad , \quad x = \varepsilon\varphi\theta \,, \; -\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$$dx = \tau\varepsilon\mu^2\theta\, d\theta$$

$$\int \frac{1}{x^2(1+x^2)}\,dx = \int \frac{1}{\varepsilon\varphi^2\theta\,(1+\varepsilon\varphi^2\theta)} \cdot \tau\varepsilon\mu^2\theta\, d\theta =$$

$$= \int \frac{(1+\varepsilon\varphi^2\theta)}{\varepsilon\varphi^2\theta\,(1+\varepsilon\varphi^2\theta)}\, d\theta \qquad \text{διότι} \quad \tau\varepsilon\mu^2\theta = 1+\varepsilon\varphi^2\theta$$

$$= \int \frac{1}{\varepsilon\varphi^2\theta}\, d\theta = \int \sigma\varphi^2\theta\, d\theta = \int (\sigma\tau\varepsilon\mu^2\theta - 1)\, d\theta$$

$$= -\sigma\varphi\theta - \theta + c$$

$$= -\frac{1}{\varepsilon\varphi\theta} - \theta + c$$

$$= -\frac{1}{x} - \tau o\xi\varepsilon\varphi\, x + c \qquad \text{διότι} \quad \theta = \tau o\xi\varepsilon\varphi\, x$$

$$\text{και} \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

(A12) $\int \frac{\sqrt{x^2-1}}{x^2}\,dx$, $x = \tau\epsilon\mu\theta$, $0<\theta<\frac{\pi}{2}$

$dx = \tau\epsilon\mu\theta \cdot \epsilon\varphi\theta\, d\theta$

$$\int \frac{\sqrt{x^2-1}}{x^2}\,dx = \int \frac{\sqrt{\tau\epsilon\mu^2\theta - 1}}{\tau\epsilon\mu^2\theta} \cdot \tau\epsilon\mu\theta \cdot \epsilon\varphi\theta\, d\theta$$

$$= \int \frac{\sqrt{\epsilon\varphi^2\theta} \cdot \epsilon\varphi\theta}{\tau\epsilon\mu\theta}\, d\theta$$

$$= \int \frac{|\epsilon\varphi\theta| \cdot \epsilon\varphi\theta}{\tau\epsilon\mu\theta}\, d\theta$$

$$= \int \frac{\epsilon\varphi^2\theta}{\tau\epsilon\mu\theta}\, d\theta$$ διότι $\epsilon\varphi\theta > 0$ στο $0<\theta<\frac{\pi}{2}$

$$= \int \frac{\tau\epsilon\mu^2\theta - 1}{\tau\epsilon\mu\theta}\, d\theta$$ διότι $\epsilon\varphi^2\theta = \tau\epsilon\mu^2\theta - 1$

$$= \int \left(\tau\epsilon\mu\theta - \frac{1}{\tau\epsilon\mu\theta}\right) d\theta$$

$$= \int \tau\epsilon\mu\theta\, d\theta - \int \sigma\upsilon\nu\theta\, d\theta$$ διότι $\sigma\upsilon\nu\theta = \frac{1}{\tau\epsilon\mu\theta}$

$$= \int \frac{\tau\epsilon\mu\theta(\tau\epsilon\mu\theta + \epsilon\varphi\theta)}{\tau\epsilon\mu\theta + \epsilon\varphi\theta} - \int \sigma\upsilon\nu\theta\, d\theta$$

$$= \int \frac{\tau\epsilon\mu^2\theta + \tau\epsilon\mu\theta\,\epsilon\varphi\theta}{\tau\epsilon\mu\theta + \epsilon\varphi\theta}\, d\theta - \int \sigma\upsilon\nu\theta\, d\theta$$

$$= \int \frac{(\tau\epsilon\mu\theta + \epsilon\varphi\theta)'}{\tau\epsilon\mu\theta + \epsilon\varphi\theta}\, d\theta - \eta\mu\theta$$

$$= \ln|\tau\epsilon\mu\theta + \epsilon\varphi\theta| - \eta\mu\theta + c$$

$$= \ln\left|x + \sqrt{x^2-1}\right| - \frac{\sqrt{x^2-1}}{x} + c$$

$\epsilon\varphi^2\theta = \tau\epsilon\mu^2\theta - 1$
$= x^2 - 1$
$\epsilon\varphi\theta = \sqrt{x^2-1}$

Επίσης $x = \frac{1}{\sigma\upsilon\nu\theta} \Rightarrow$
$\sigma\upsilon\nu\theta = \frac{1}{x}$
$\eta\mu\theta = \sqrt{1 - \sigma\upsilon\nu^2\theta}$
$= \sqrt{1 - \frac{1}{x^2}} = \sqrt{\frac{x^2-1}{x^2}}$

INTGEN-A13

(A13) $\int \eta\mu(\ln x)\,dx$

$$u = \ln x \Rightarrow x = e^u$$
$$dx = e^u\,du$$

$$\int \eta\mu(\ln x)\,dx = \int \eta\mu u \cdot e^u\,du = I$$

$$I = \int \eta\mu u\, d(e^u)$$
$$= e^u \eta\mu u - \int e^u\, d(\eta\mu u)$$
$$= e^u \eta\mu u - \int e^u \sigma\upsilon\nu u\, du$$
$$= e^u \eta\mu u - \int \sigma\upsilon\nu u\, d(e^u)$$
$$= e^u \eta\mu u - \left[e^u \sigma\upsilon\nu u - \int e^u\, d(\sigma\upsilon\nu u)\right]$$
$$= e^u \eta\mu u - e^u \sigma\upsilon\nu u + \int e^u\, d(\sigma\upsilon\nu u)$$
$$I = e^u \eta\mu u - e^u \sigma\upsilon\nu u - \int e^u \eta\mu u\, du$$
$$I = e^u \eta\mu u - e^u \sigma\upsilon\nu u - I$$
$$\Rightarrow 2I = e^u \eta\mu u - e^u \sigma\upsilon\nu u$$
$$I = \frac{1}{2}\left(e^u \eta\mu u - e^u \sigma\upsilon\nu u\right) + c$$
$$I = \frac{1}{2}\left[x\,\eta\mu(\ln x) - x\,\sigma\upsilon\nu(\ln x)\right] + c$$

(A14) $\displaystyle\int \frac{1}{5+3\sigma\upsilon\nu x}\,dx$

$t = \varepsilon\varphi\frac{x}{2}$

$$\eta\mu x = \frac{2\varepsilon\varphi\frac{x}{2}}{1+\varepsilon\varphi^2\frac{x}{2}} = \frac{2t}{1+t^2}$$

$$\sigma\upsilon\nu x = \frac{1-\varepsilon\varphi^2\frac{x}{2}}{1+\varepsilon\varphi^2\frac{x}{2}} = \frac{1-t^2}{1+t^2}$$

$$\frac{x}{2} = \tau o\xi\varepsilon\varphi\, t$$

$$x = 2\,\tau o\xi\varepsilon\varphi\, t$$

$$dx = 2\,\frac{1}{1+t^2}\,dt$$

Άρα:

$$\int \frac{1}{5+3\sigma\upsilon\nu x}\,dx = \int \frac{1}{5+\frac{3(1-t^2)}{1+t^2}}\cdot 2\,\frac{1}{1+t^2}\,dt =$$

$$= \int \frac{2}{\frac{5+5t^2+3-3t^2}{1+t^2}}\cdot\frac{1}{1+t^2}\,dt$$

$$= \int \frac{2(1+t^2)\,dt}{(2t^2+8)(1+t^2)} = \int \frac{1}{t^2+4}\,dt$$

$$= \int \frac{1}{4\left[\left(\frac{t}{2}\right)^2+1\right]}\,dt = \frac{1}{4}\int \frac{2\,d\left(\frac{t}{2}\right)}{\left(\frac{t}{2}\right)^2+1}$$

$$= \frac{1}{2}\,\tau o\xi\varepsilon\varphi\,\frac{t}{2} + c$$

$$= \frac{1}{2}\,\tau o\xi\varepsilon\varphi\left(\frac{\varepsilon\varphi\frac{x}{2}}{2}\right) + c$$

(A15) $\displaystyle\int \frac{1}{\eta\mu^2 x\,(1-\varepsilon\varphi x)}\,dx$, $t=\varepsilon\varphi x$

$x = \tau o \xi \varepsilon \varphi\, t$

$dx = \dfrac{1}{1+t^2}\,dt$

Άρα

$\sigma\upsilon\nu 2x = \dfrac{1-t^2}{1+t^2}$

$$\int \frac{1}{\eta\mu^2 x\,(1-\varepsilon\varphi x)}\,dx = \int \frac{1}{\frac{1-\sigma\upsilon\nu 2x}{2}(1-t)}\,\frac{dt}{1+t^2} =$$

$$= \int \frac{2}{\left(1-\frac{1-t^2}{1+t^2}\right)(1-t)}\cdot\frac{dt}{1+t^2}$$

$$= \int \frac{2\,dt}{\frac{(1+t^2-1+t^2)\cdot(1-t)}{1+t^2}(1+t^2)} = \int \frac{dt}{t^2(1-t)} =$$

$$= \int\left(\frac{1}{t}+\frac{1}{t^2}+\frac{1}{1-t}\right)dt$$

$$= \int \frac{1}{t}\,dt - \int \frac{-1}{1-t}\,dt + \int \frac{1}{t^2}\,dt$$

$$= \ln|t| - \ln|1-t| - \frac{1}{t} + c$$

$$= \ln\left|\frac{t}{1-t}\right| - \frac{1}{t} + c$$

$$= \ln\left|\frac{\varepsilon\varphi x}{1-\varepsilon\varphi x}\right| - \frac{1}{\varepsilon\varphi x} + c$$

$$= \ln\left|\frac{\varepsilon\varphi x}{1-\varepsilon\varphi x}\right| - \sigma\varphi x + c$$

Αναλύουμε το κλάσμα $\dfrac{1}{t^2(1-t)}$ σε άθροισμα απλών κλασμάτων:

$$\frac{1}{t^2(1-t)} = \frac{A}{t} + \frac{B}{t^2} + \frac{\Gamma}{1-t}$$

$$1 \equiv A\cdot t(1-t) + B(1-t) + \Gamma\cdot t^2$$

Για $t=1 \Rightarrow 1=\Gamma\cdot 1 \Rightarrow \Gamma=1$

Για $t=0 \Rightarrow 1=B \Rightarrow B=1$

Για $t=-1 \Rightarrow 1=-2A+2B+\Gamma$

$A=1$

Άρα

$$\frac{1}{t^2(1-t)} = \frac{1}{t} + \frac{1}{t^2} + \frac{1}{1-t}$$

(A16) $\int \frac{1}{3+2\sigma\upsilon\nu x-\eta\mu x}dx \quad , \quad t=\varepsilon\varphi\frac{x}{2}$

$$\frac{x}{2}=\tau o\xi\,\varepsilon\varphi\, t$$

$$x=2\,\tau o\xi\,\varepsilon\varphi\, t$$

$$dx=2\frac{1}{1+t^2}dt$$

$$\eta\mu x=\frac{2\varepsilon\varphi\frac{x}{2}}{1+\varepsilon\varphi^2\frac{x}{2}}=\frac{2t}{1+t^2}$$

$$\sigma\upsilon\nu x=\frac{1-\varepsilon\varphi^2\frac{x}{2}}{1+\varepsilon\varphi^2\frac{x}{2}}=\frac{1-t^2}{1+t^2}$$

Άρα

$$\int\frac{1}{3+2\sigma\upsilon\nu x-\eta\mu x}dx=\int\frac{1}{3+\frac{2(1-t^2)}{1+t^2}-\frac{2t}{1+t^2}}\cdot\frac{2}{1+t^2}dt$$

$$=\int\frac{2dt}{\frac{3+3t^2+2-2t^2-2t}{1+t^2}\cdot(1+t^2)}$$

$$=\int\frac{2dt}{t^2-2t+1+4}=\int\frac{2dt}{(t-1)^2+4}=\frac{1}{4}\int\frac{2dt}{\frac{(t-1)^2}{4}+1}$$

$$=\frac{1}{2}\int\frac{dt}{\left(\frac{t-1}{2}\right)^2+1}=\frac{1}{2}\int\frac{2\,d\left(\frac{t-1}{2}\right)}{\left(\frac{t-1}{2}\right)^2+1}$$

$$=\tau o\xi\,\varepsilon\varphi\left(\frac{t-1}{2}\right)+c$$

$$=\tau o\xi\,\varepsilon\varphi\left(\frac{\varepsilon\varphi\frac{x}{2}-1}{2}\right)+c$$

(A17) $\int e^{\sqrt{x}}dx$, $u=\sqrt{x}$

$u^2 = x$

$2u\,du = dx$

Άρα

$$\int e^{\sqrt{x}}dx = \int e^u \cdot 2u\,du$$
$$= 2\int u\,d(e^u)$$
$$= 2\left[ue^u - \int e^u du\right]$$
$$= 2ue^u - 2e^u + c$$
$$= 2\left(\sqrt{x}\,e^{\sqrt{x}} - e^{\sqrt{x}}\right) + c$$

(A18) $\int \frac{9}{e^{2x}-3e^x}dx$ $\quad u=e^x$

$du = e^x dx \Rightarrow dx = \frac{du}{e^x} = \frac{du}{u}$

$$\int \frac{9}{e^{2x}-3e^x}dx = \int \frac{9}{(e^x)^2-3e^x}dx$$
$$= \int \frac{9}{u^2-3u}\cdot\frac{du}{u} = \int \frac{9\,du}{u^2(u-3)}$$
$$= \int\left(-\frac{1}{u} - \frac{3}{u^2} + \frac{1}{u-3}\right)du$$
$$= -\ln|u| + \frac{3}{u} + \ln|u-3| + c$$
$$= -\ln e^x + \frac{3}{e^x} + \ln|e^x-3| + c$$
$$= -x + \frac{3}{e^x} + \ln|e^x-3| + c$$

Αναλύουμε το κλάσμα $\frac{9}{u^2(u-3)}$ σε άθροισμα απλών κλασμάτων:

$$\frac{9}{u^2(u-3)} = \frac{A}{u} + \frac{B}{u^2} + \frac{\Gamma}{u-3}$$

$$9 \equiv Au(u-3) + B(u-3) + \Gamma \cdot u^2$$

$u=0 \Rightarrow 9=-3B \Rightarrow B=-3$

$u=3 \Rightarrow 9=9\Gamma \Rightarrow \Gamma=1$

$u=4 \Rightarrow 9=4A+B+16\Gamma$

$\Rightarrow 9 = 4A-3+16$

$\Rightarrow A=-1$

$$\frac{9}{u^2(u-3)} = -\frac{1}{u} - \frac{3}{u^2} + \frac{1}{u-3}$$