Λύσεις παραβολή σελ. 60

A' ομάδα

(1) (α) $y^2=2x \Rightarrow 2yy'=2 \Rightarrow y'=\frac{1}{y} \Rightarrow \lambda_{εφ(8,4)}=\frac{1}{4}$

$\Rightarrow \lambda_{κ(8,4)}=-4 \Rightarrow$ εξίσωση εφαπτομένης

$y-4=\frac{1}{4}(x-8) \Rightarrow 4y-16=x-8$

$\Rightarrow \boxed{x-4y+8=0}$

εξίσωση καθέτου: $y-4=-4(x-8) \Rightarrow y-4=-4x+32$

$\Rightarrow \boxed{y+4x=36}$

(β) $y^2=5x \Rightarrow 2yy'=5 \Rightarrow y'=\frac{5}{2y} \Rightarrow \lambda_{εφ(0,0)}=\pm\infty$ (δεν ορίζεται)

$\Rightarrow$ εξίσωση εφαπτομένης στο $(0,0)$: $\boxed{x=0}$ και εξίσωση κάθετης $\boxed{y=0}$

(γ) $x^2=-9y \Rightarrow 2x=-9y' \Rightarrow y'=\frac{2x}{-9} \Rightarrow \lambda_{εφ(6,-4)}=\frac{12}{-9}$

$\Rightarrow \lambda_{εφ}=-\frac{4}{3} \Rightarrow \lambda_{κ}=\frac{3}{4}$

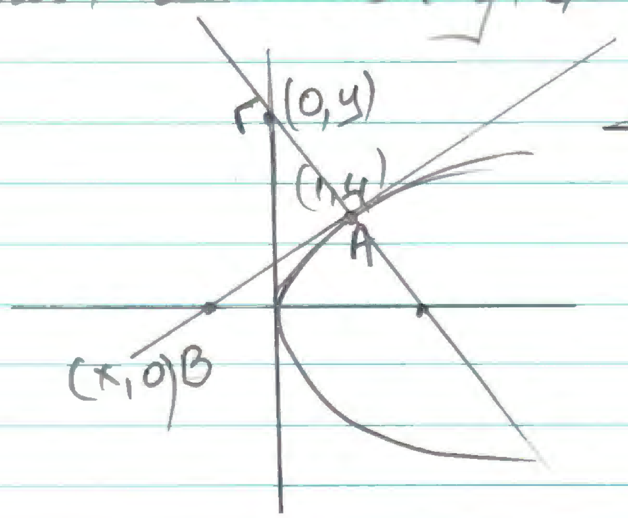
εξίσωση εφαπτομένης: $y+4=-\frac{4}{3}(x-6) \Rightarrow 3y+12=-4x+24$

$\Rightarrow \boxed{3y+4x-12=0}$

εξίσωση κάθετης: $y+4=\frac{3}{4}(x-6) \Rightarrow 4y+16=3x-18$

$\Rightarrow \boxed{3x-4y=34}$

(2)



$y^2=16x$

$\Rightarrow 2yy'=16 \Rightarrow y'=\frac{8}{y}$

$\Rightarrow \lambda_{εφ(1,4)}=2$ και $\lambda_{κ(1,4)}=-\frac{1}{2}$

εξίσωση εφαπτομένης στο A: $y-4=2(x-1)$

$\Rightarrow y-4=2x-2$

$\Rightarrow \boxed{y-2x=2}$ για $y_B=0 \Rightarrow x=-1$

$\Rightarrow \boxed{B(-1,0)}$

εξίσωση κάθετης στο A: $y-4=-\frac{1}{2}(x-1) \Rightarrow 2y-8=-x+1$

$\Rightarrow \boxed{2y+x=9}$ για $x_\Gamma=0 \Rightarrow \boxed{y=\frac{9}{2}}$

$\Rightarrow \Gamma=\left(0,\frac{9}{2}\right)$

$$\Rightarrow E=\frac{1}{2}\left|\begin{vmatrix} 1 & 4 & 1 \\ -1 & 0 & 1 \\ 0 & \frac{9}{2} & 1 \end{vmatrix}\right| = \frac{1}{2}\left|-\frac{9}{2}-\frac{9}{2}+4\right| = \frac{1}{2}\cdot 5 = \frac{5}{2} \text{ τ.μ.}$$

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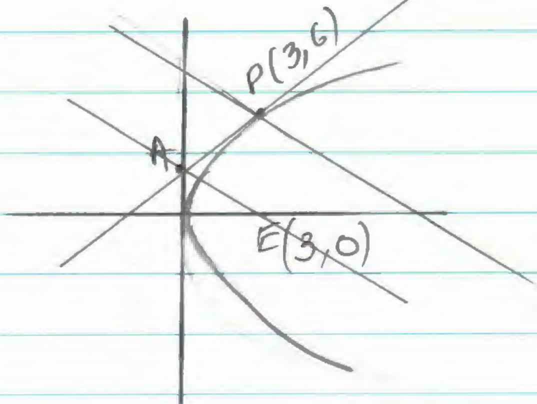
(3)



$y^2=12x$

$2yy'=12 \Rightarrow y'=\frac{6}{y}$

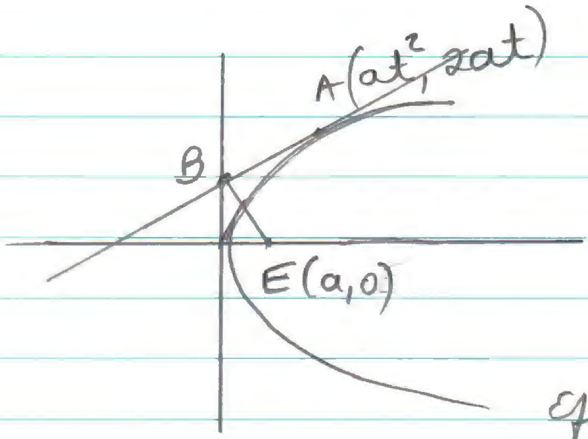
$\Rightarrow \lambda_{εφ_P}=\frac{6}{6}=1 \Rightarrow \lambda_{κ_P}=-1$

εξίσωση της εφαπτομένης στο A: $y-6=1(x-3) \Rightarrow \boxed{y-x=3}$

$x_A=0 \Rightarrow y=3 \Rightarrow A(0,3) \Rightarrow \lambda_{AE}=\frac{0-3}{3-0}=\frac{-3}{3}=-1$

$\Rightarrow \lambda_{AE}=\lambda_{κ_P} \Rightarrow AE // κ_P.$

(4)



$y^2 = 4ax \Rightarrow 2yy' = 4a$
$\Rightarrow \boxed{y' = \frac{2a}{y}}$

$\Rightarrow \lambda_{\varepsilon\varphi_A} = \frac{2a}{2at} = \frac{1}{t}$

εξίσωση εφαπτομένης στο A:

$y - 2at = \frac{1}{t}(x - at^2)$

$\Rightarrow yt - 2at^2 = x - at^2 \Rightarrow \boxed{yt = x + at^2}$

$x_B = 0 \Rightarrow y = at \Rightarrow B(0, at)$

$\Rightarrow \lambda_{BE} = \frac{at - 0}{0 - a} = -t \Rightarrow \lambda_{\varepsilon\varphi_A} \lambda_{BE} = \frac{1}{t}(-t) = -1$

$\Rightarrow \boxed{\hat{ABE} = 90^\circ}$

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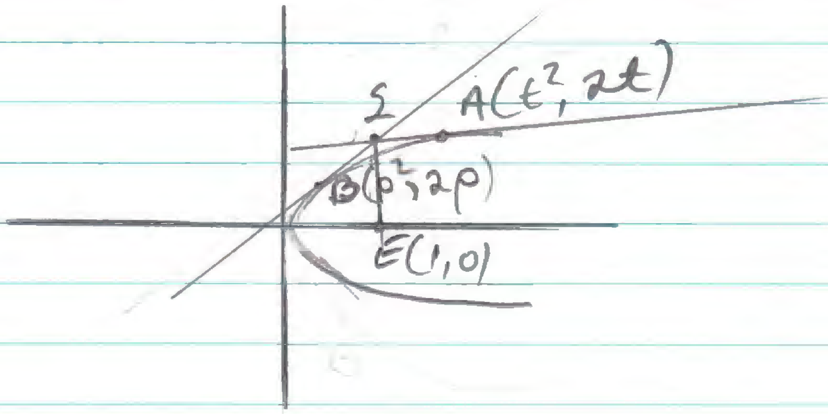
(5) $y^2 = 4x$



$2yy' = 4 \Rightarrow y' = \frac{2}{y}$

$\lambda_{\varepsilon\varphi_A} = \frac{2}{2t} = \frac{1}{t}$

$\lambda_{\varepsilon\varphi_B} = \frac{2}{2p} = \frac{1}{p}$

εξίσωση εφαπτομένης στο A: $y - 2t = \frac{1}{t}(x - t^2) \Rightarrow yt - 2t^2 = x - t^2$

$\Rightarrow \boxed{yt - x = t^2}$

Όμοια η εξίσωση της εφαπτομένης στο B: $\boxed{yp - x = p^2}$

Εύρεση Σ: $\left.\begin{array}{l} yt - x = t^2 \\ yp - x = p^2 \end{array}\right\} \Rightarrow yt - yp = t^2 - p^2$

$\Rightarrow y = \frac{(t-p)(t+p)}{t-p} = t + p$

$\Rightarrow (t+p)t - x = t^2 \Rightarrow t^2 + pt - x = t^2 \Rightarrow \boxed{x = pt}$

$\Rightarrow \Sigma(pt,\ t+p)$ αφού $pt = 1 \Rightarrow \Sigma(1, t+p)$

$\Rightarrow \lambda_{\Sigma E} = \dfrac{t+p-0}{1-0} = \infty$ (Δεν ορίζεται)

$\Rightarrow$ εξίσωση ΣΕ : $\boxed{x = 1}$

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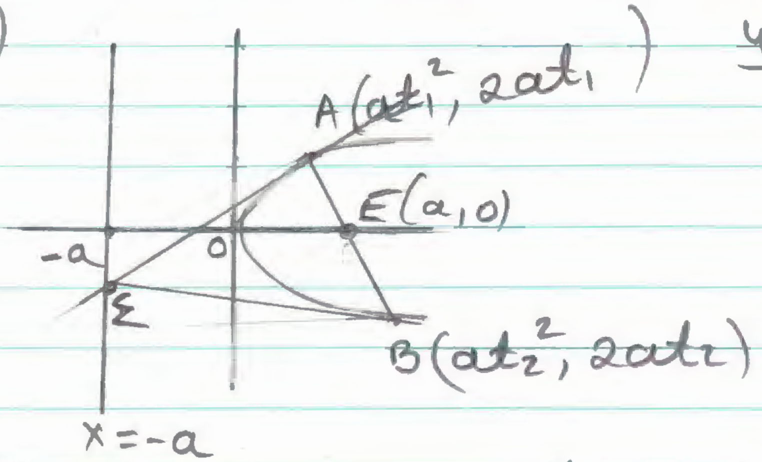
(6)



$y^2 = 4ax$

$\lambda_{AE} = \dfrac{2at_1 - 0}{at_1^2 - a} = \dfrac{2t_1}{t_1^2 - 1}$

$\lambda_{BE} = \dfrac{2at_2 - 0}{at_2^2 - a} = \dfrac{2t_2}{t_2^2 - 1}$

(a)

$\lambda_{AE} = \lambda_{BE} \Rightarrow \dfrac{\cancel{2}t_1}{t_1^2 - 1} = \dfrac{\cancel{2}t_2}{t_2^2 - 1} \Rightarrow t_1 t_2^2 - t_1 = t_2 t_1^2 - t_2$

$\Rightarrow t_1 t_2 (t_2 - t_1) + (t_2 - t_1) = 0 \Rightarrow (t_2 - t_1)(t_1 t_2 + 1) = 0$

$\Rightarrow$ $t_2 = t_1$ απορ. $\boxed{t_1 t_2 = -1}$

(β) $2yy' = 4a \Rightarrow y' = \dfrac{2a}{y} \Rightarrow \lambda_{\varepsilon\varphi_A} = \dfrac{1}{t_1}$, $\lambda_{\varepsilon\varphi_B} = \dfrac{1}{t_2}$

Εξίσωση εφαπτομένης στο A: $y - 2at_1 = \dfrac{1}{t_1}(x - at_1^2)$

$\Rightarrow yt_1 - 2at_1^2 = x - at_1^2 \Rightarrow \boxed{yt_1 - x = at_1^2}$

όμοια η εξίσωση της εφαπτ. στο B: $\boxed{yt_2 - x = at_2^2}$

$$\lambda_{\varepsilon\varphi_A} \cdot \lambda_{\varepsilon\varphi_B} = \frac{1}{t_1} \cdot \frac{1}{t_2} = \frac{1}{t_1 t_2} = \frac{1}{-1} = -1 \Rightarrow \varepsilon\varphi_A \perp \varepsilon\varphi_B$$

Εύρεση του Σ: $\left.\begin{array}{l} y t_1 - x = a t_1^2 \\ y t_2 - x = a t_2^2 \end{array}\right| \begin{array}{l} t_2 \\ -t_1 \end{array}$

$$\Rightarrow \begin{array}{l} \cancel{y t_1 t_2} - x t_2 = a t_1^2 t_2 \\ \cancel{-y t_2 t_1} + x t_1 = -a t_2^2 t_1 \end{array}$$

$$x(t_1 - t_2) = a t_1 t_2 (t_1 - t_2)$$

$$\left.\begin{array}{l} \Rightarrow x = a t_1 t_2 \\ t_1 t_2 = -1 \end{array}\right\} \Rightarrow \boxed{x = -a}$$

$\Rightarrow$ Σ σημείο της διευθετούσας.

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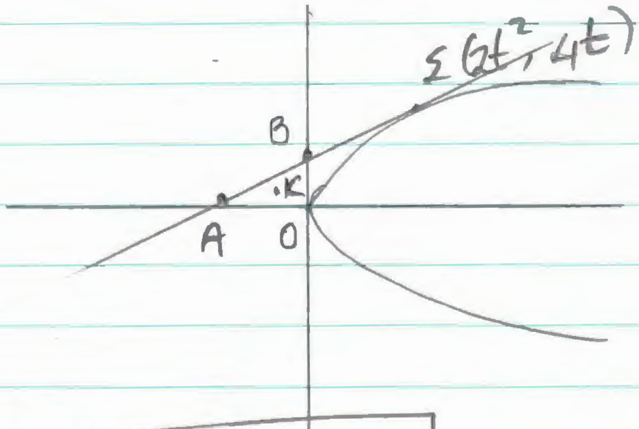
7 $y^2 = 8x$



$2yy' = 8 \Rightarrow y' = \frac{4}{y}$

$\Rightarrow \lambda_{\varepsilon\varphi_Σ} = \frac{4}{4t} = \frac{1}{t}$

$\Rightarrow y - 4t = \frac{1}{t}(x - 2t^2)$

$yt - 4t^2 = x - 2t^2 \Rightarrow \boxed{yt - x = 2t^2}$

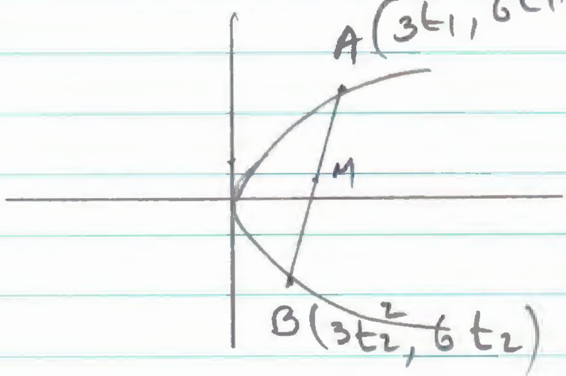
για $x = 0 \Rightarrow y = 2t \Rightarrow B(0, 2t)$

για $y = 0 \Rightarrow x = -2t^2 \Rightarrow A(-2t^2, 0)$

$$\Rightarrow \left\{\begin{array}{l} x_K = \frac{0 + (-2t^2) + 0}{3} = -\frac{2t^2}{3} \\ y_K = \frac{0 + 0 + 2t}{3} = \frac{2t}{3} \Rightarrow t = \frac{3y}{2} \end{array}\right\} \Rightarrow x = -\frac{2}{3}\left(\frac{3y}{2}\right)^2$$

$$\boxed{x = -\frac{3y^2}{2}}$$

(8)



$y^2 = 12x$

$$\left.\begin{aligned} x_M &= \frac{3t_1^2+3t_2^2}{2} \\ y_M &= \frac{6t_1+6t_2}{2} = 3t_1+3t_2 \end{aligned}\right\} \begin{aligned} 2x &= 3(t_1^2+t_2^2) \\ y &= 3(t_1+t_2) \end{aligned}$$

(α) $t_1^2+t_2^2 = 3t_1t_2$

$$y^2 = 9(t_1^2+t_2^2+2t_1t_2) \Rightarrow y^2 = 9\left(t_1^2+t_2^2+\frac{2}{3}(t_1^2+t_2^2)\right)$$

$$\Rightarrow y^2 = 9\left(\frac{2x}{3}+\frac{2}{3}\cdot\frac{2x}{3}\right) \Rightarrow y^2 = 6x+4x \Rightarrow \boxed{y^2 = 10x}$$

(β) $\lambda_{AB} = \text{εφ}45 \Rightarrow \dfrac{6t_1-6t_2}{3t_1^2-3t_2^2} = 1 \Rightarrow \dfrac{6\cancel{(t_1-t_2)}}{3\cancel{(t_1-t_2)}(t_1+t_2)} = 1$

$$\Rightarrow \frac{2}{t_1+t_2} = 1 \Rightarrow \boxed{t_1+t_2 = 2} \Rightarrow y = 3\cdot 2 \Rightarrow \boxed{y=6}$$

(γ) $\lambda_{AM} = \lambda_{BM} \xrightarrow{\text{αποδείχθηκε στην 6.}} t_1t_2 = -1$

$$y^2 = 9(t_1+t_2)^2 \Rightarrow y^2 = 9(t_1^2+2t_1t_2+t_2^2)$$

$$\Rightarrow y^2 = 9\left(\frac{2x}{3}+2(-1)\right) \Rightarrow \boxed{y^2 = 6x-18}$$

(δ) $\lambda_{AB} = \dfrac{6t_1 - 6t_2}{3t_1^2 - 3t_2^2} = \dfrac{2}{t_1 + t_2}$

εξίσωση AB: $y - 6t_1 = \dfrac{2}{t_1+t_2}(x - 3t_1^2)$

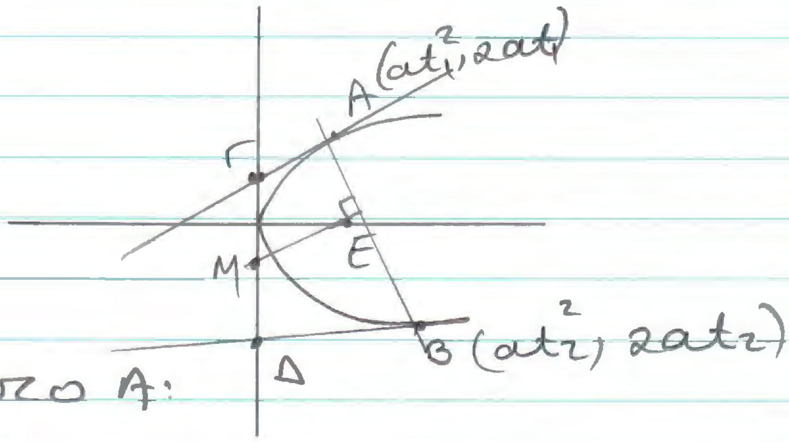
$\Rightarrow y(t_1+t_2) - \cancel{6t_1^2} - 6t_1t_2 = 2x - \cancel{6t_1^2}$

$\Rightarrow \boxed{y(t_1+t_2) = 2x + 6t_1t_2}$ $(0,6) \Rightarrow 6(t_1+t_2) = 6t_1t_2$

$\Rightarrow \boxed{t_1+t_2 = t_1t_2}$ $\Rightarrow y^2 = 9(t_1^2 + 2t_1t_2 + t_2^2)$

$\Rightarrow y^2 = 9\left(\dfrac{2x}{3} + 2\dfrac{y}{3}\right) \Rightarrow \boxed{y^2 = 6x + 6y}$

(9) $y^2 = 4ax$



$2yy' = 4a \Rightarrow y' = \dfrac{2a}{y}$

$\Rightarrow \lambda_{εφ_A} = \dfrac{2a}{2at_1} = \dfrac{1}{t_1}$

εξίσωση εφαπτομένης στο A:

$y - 2at_1 = \dfrac{1}{t_1}(x - at_1^2) \Rightarrow yt_1 - 2at_1^2 = x - at_1^2$

$\Rightarrow \boxed{yt_1 - x = at_1^2}$

όμοια εξίσωση εφαπτομένης στο B: $\boxed{yt_2 - x = at_2^2}$

$x_Γ = 0 \Rightarrow yt_1 = at_1^2 \Rightarrow y = at_1$ $Γ(0, at_1)$

$x_Δ = 0 \Rightarrow yt_2 = at_2^2 \Rightarrow y = at_2$ $Δ(0, at_2)$

$x_M = 0$ $\qquad$ $M\left(0, \frac{at_1+at_2}{2}\right)$

$y_M = \frac{at_1+at_2}{2}$ $\qquad$ $E(a,0)$

$$\lambda_{ME} = \frac{\frac{at_1+at_2}{2} - 0}{0 - a} = -\frac{(t_1+t_2)}{2}$$

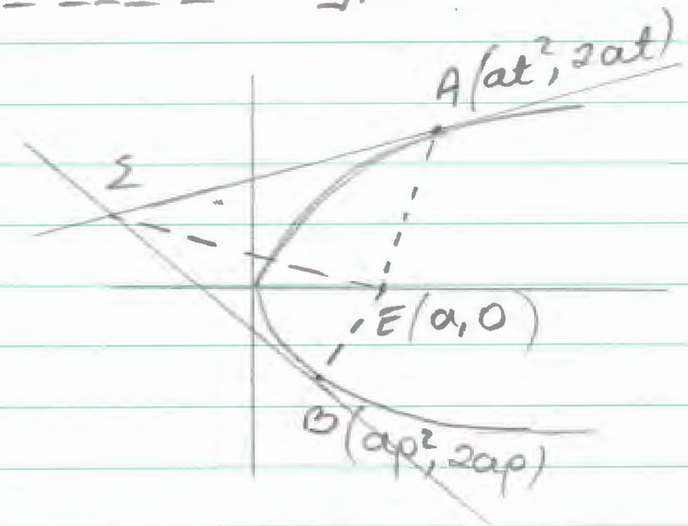
$$\lambda_{AB} = \frac{2at_1 - 2at_2}{at_1^2 - at_2^2} = \frac{2\cancel{a}\cancel{(t_1-t_2)}}{\cancel{a}\cancel{(t_1-t_2)}(t_1+t_2)} = \frac{2}{t_1+t_2}$$

$$\Rightarrow \lambda_{ME} \cdot \lambda_{AB} = -\frac{\cancel{(t_1+t_2)}}{\cancel{2}} \cdot \frac{\cancel{2}}{\cancel{t_1+t_2}} = -1$$

$$\Rightarrow \boxed{ME \perp AB}$$

Ομάδα Β σελ. 62

(10)



$$y^2=4ax$$

$$2yy'=4a \Rightarrow y'=\frac{2a}{y}$$

$$\Rightarrow \lambda_{εφ_A}=\frac{2a}{2at}=\frac{1}{t}$$

$$\lambda_{εφ_B}=\frac{1}{p}$$

Εξίσωση εφαπτομένης στο Α:

$$y-2at=\frac{1}{t}(x-at^2) \Rightarrow yt-2at^2=x-at^2$$

$$\Rightarrow \boxed{x-yt+at^2=0} \quad (1)$$

Εξίσωση εφαπτομένης στο Β: $\boxed{x-yp+ap^2=0}$ (2)

Εύρεση Σ: Λύνω σύστημα (1)+(2)

$$\begin{aligned} x-yt+at^2&=0\\ -x+yp-ap^2&=0 \end{aligned}$$

$$y(p-t)+a(t^2-p^2)=0 \Rightarrow y(p-t)-a(p-t)(p+t)=0$$

$$\Rightarrow y=\frac{a\cancel{(p-t)}(p+t)}{\cancel{p-t}} \Rightarrow \boxed{y=a(p+t)}$$

$$\Rightarrow x-a(p+t)\cdot t+at^2=0 \Rightarrow x-apt\cancel{-at^2}\cancel{+at^2}=0$$

$$\Rightarrow \boxed{x=apt}$$

$$\Rightarrow Σ\left(apt,\ a(p+t)\right)$$

$$(SE)^2 = (apt-a)^2 + [a(p+t)-0]^2$$

$$\Rightarrow \boxed{SE^2 = a^2(pt-1)^2 + a^2(p+t)^2}$$

$$(AE) = \sqrt{(at^2-a)^2 + (2at-0)^2} = \sqrt{a^2(t^2-1)^2 + 4a^2t^2}$$

$$(BE) = \sqrt{(ap^2-a)^2 + (2ap-0)^2} = \sqrt{a^2(p^2-1) + 4a^2p^2}$$

$$\Rightarrow (AE)(BE) = \sqrt{a^2[(t^2-1)^2 + 4t^2] \cdot a^2[(p^2-1) + 4p^2]} =$$

$$= a^2\sqrt{(t^4-2t^2+1+4t^2) \cdot (p^4-2p^2+1+4p^2)}$$

$$= a^2\sqrt{(t^4+2t^2+1)(p^4+2p^2+1)^2}$$

$$= a^2\sqrt{(t^2+1)^2(p^2+1)^2} = a^2(t^2+1)(p^2+1) \quad (3)$$

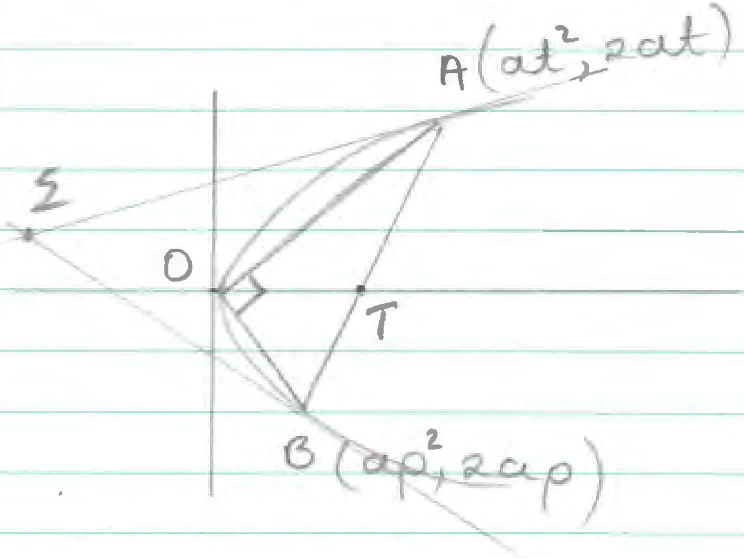
$$(SE)^2 = a^2(p^2t^2 - \cancel{2pt} + 1 + p^2 + \cancel{2pt} + t^2)$$

$$= a^2(p^2t^2 + p^2 + 1 + t^2) = a^2[p^2(t^2+1) + (1+t^2)]$$

$$= a^2(t^2+1)(p^2+1) \quad (4)$$

$$\Rightarrow \text{από } (3) + (4) \quad \boxed{(SE)^2 = (AE)(BE)}$$

11



$y^2 = 4ax$

$\lambda_{OB} \cdot \lambda_{OA} = -1$

$\lambda_{OB} = \dfrac{2at - 0}{at^2 - 0} = \dfrac{2}{t}$

$\lambda_{OA} = \dfrac{2ap - 0}{ap^2 - 0} = \dfrac{2}{p}$

(a) $\Rightarrow \lambda_{OB} \cdot \lambda_{OA} = -1 \Rightarrow \dfrac{2}{t} \cdot \dfrac{2}{p} = -1 \Rightarrow \boxed{tp = -4}$

Εξίσωση AB: $\lambda_{AB} = \dfrac{2ap - 2at}{ap^2 - at^2} = \dfrac{2(p-t)}{(p-t)(p+t)} = \dfrac{2}{p+t}$

$\Rightarrow y - 2at = \dfrac{2}{p+t}(x - at^2)$

$\Rightarrow yp + yt - 2atp - 2at^2 = 2x - 2at^2$

$\Rightarrow \boxed{yp + yt - 2atp = 2x}$

$y_T = 0$

$\Rightarrow \left.\begin{matrix} x = -atp \\ tp = -4 \end{matrix}\right\} \Rightarrow \boxed{x = +4a}$

$\Rightarrow T(4a, 0)$ σταθερό σημείο.

(β) $2yy' = 4a \Rightarrow y' = \dfrac{2a}{y} \Rightarrow \lambda_{εφ_A} = \dfrac{2a}{2at} = \dfrac{1}{t}$

$\lambda_{εφ_B} = \dfrac{1}{p}$

Εξίσωση εφαπτομένης στο A: $y - 2at = \dfrac{1}{t}(x - at^2)$

$$yt - 2at^2 = x - at^2 \Rightarrow \boxed{yt - at^2 = x} \quad (1)$$

εξίσωση εφαπτομένης στο Β: $\boxed{yp - ap^2 = x}$ (2)

Εύρεση Σ: λύνω σύστημα (1) + (2)

$$yt - at^2 = yp - ap^2 \Rightarrow y(t-p) - a(t^2 - p^2) = 0$$

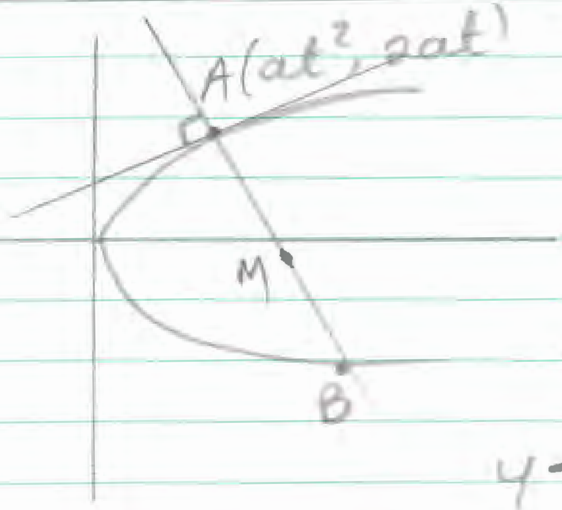
$$\Rightarrow y(t-p) = a(t-p)(t+p) \Rightarrow \boxed{y = a(t+p)}$$

$$\Rightarrow a(t+p)t - at^2 = x \Rightarrow at^2 + apt - at^2 = x$$

$$\Rightarrow x = apt \Rightarrow Σ\left[apt, a(t+p)\right]$$

$$\left.\begin{array}{l} x = apt \\ pt = -4 \end{array}\right\} \Rightarrow \boxed{x = -4a} \text{ γ.τ του } Σ.$$

12



$$y^2 = 4ax \quad (1)$$

$$λε_{φA} = \frac{1}{t} \Rightarrow λ_{κA} = -t$$

$\Rightarrow$ εξίσωση καθέτου στο A:

$$y - 2at = -t(x - at^2)$$

$$\Rightarrow \boxed{y - 2at = -tx + at^3} \quad (2)$$

Εύρεση Β: λύνω σύστημα (1) + (2)

$$y - 2at = -t \cdot \frac{y^2}{4a} + at^3 \Rightarrow y4a - 8a^2t = -ty^2 + 4a^2t^3$$

$$ty^2 + 4ay - 8a^2t - 4a^2t^3 = 0$$

$$\Delta = (4a)^2 - 4 \cdot t\left(-8a^2t - 4a^2t^3\right) = 16a^2 + 32a^2t^2 + 16a^2t^4$$

$$= 16a^2\left(1 + 2t^2 + t^4\right) = 16a^2\left(1+t^2\right)^2$$

$$\Rightarrow y_{1,2} = \frac{-4a \pm \sqrt{16a^2(1+t^2)^2}}{2t} = \frac{-4a \pm 4a(1+t^2)}{2t}$$

$$y_1 = \frac{-4a + 4a + 4at^2}{2t} = 2at \Rightarrow \boxed{y_1 = y_A}$$

$$y_2 = \frac{-4a - 4a - 4at^2}{2t} = \frac{-8a - 4at^2}{2t} = \frac{-4a - 2at^2}{t}$$

$$\Rightarrow \boxed{y_B = \frac{-4a - 2at^2}{t}} \Rightarrow x = \frac{y^2}{4a} = \frac{\left[\frac{-2a(2+t^2)}{t}\right]^2}{4a}$$

$$= \frac{\frac{4a^2(2+t^2)^2}{t^2}}{4a} = \frac{a(2+t^2)^2}{t^2} \Rightarrow B\left(\frac{a(2+t^2)^2}{t^2}, -\frac{2a(2+t^2)}{t}\right)$$

$$x_M = \frac{x_A + x_B}{2} = \frac{at^2 + \frac{a(2+t^2)^2}{t^2}}{2} = \frac{at^4 + a(4 + 2t^2 + t^4)}{2t^2}$$

$$= \frac{a(t^4 + 4 + 4t^2 + t^4)}{2t^2} = \frac{\cancel{2}a(t^4 + 2 + 2t^2)}{\cancel{2}t^2} = \frac{a(t^4 + 2 + 2t^2)}{t^2}$$

$$y_M = \frac{2at + \frac{-4a - 2at^2}{t}}{2} = \frac{\cancel{2at^2} - 4a - \cancel{2at^2}}{2t} = -\frac{2a}{t}$$

$$\Rightarrow M\left(\frac{a(t^4 + 2 + 2t^2)}{t^2}, -\frac{2a}{t}\right) \quad //$$