ΠΡΟΤΕΙΝΟΜΕΝΕΣ ΛΥΣΕΙΣ

1. $\left.\begin{array}{l} x=t^3-1 \\ y=t^5+1 \end{array}\right\} \Rightarrow \dfrac{dy}{dx}=\dfrac{\frac{dy}{dt}}{\frac{dy}{dt}}=\dfrac{5t^4}{3t^2}=\dfrac{5}{3}t^2$, $\qquad \dfrac{d^2y}{dx^2}=\dfrac{d}{dt}\left(\dfrac{5}{3}t^2\right)\cdot\dfrac{dt}{dx}=\dfrac{10}{3}t\cdot\dfrac{1}{3t^2}=\dfrac{10}{9t}$.

2. $f(x)=\dfrac{x-a}{x^2-3x+2}$, $x\in\mathbb{R}-\{1,2\}$, $a\in\mathbb{R}$. $\quad \dfrac{dy}{dx}=\dfrac{x^2-3x+2-(x-a)(2x-3)}{\left(x^2-3x+2\right)^2} \Rightarrow$

$\dfrac{dy}{dx}=\dfrac{-x^2+2ax+2-3a}{\left(x^2-3x+2\right)^2}$. Ακρότατο για χ=0 $\Rightarrow \dfrac{dy}{dx}=0 \Rightarrow 2-3a=0 \Leftrightarrow a=\dfrac{2}{3}$

3. $\tau o\xi\varepsilon\phi\dfrac{\pi}{x+1}+\tau o\xi\varepsilon\phi\dfrac{1}{1+2\chi}=\dfrac{\pi}{4}$, $\quad \chi>0$. (1)

Θέτω $\tau o\xi\varepsilon\phi\dfrac{\pi}{x+1}=a \Leftrightarrow \varepsilon\phi a=\dfrac{\pi}{x+1}$, $0<a<\dfrac{\pi}{4}$ και

$\tau o\xi\varepsilon\phi\dfrac{1}{1+2\chi}=\beta \Leftrightarrow \varepsilon\phi\beta=\dfrac{1}{1+2\chi}$, $0<\beta<\dfrac{\pi}{4}$

Από (1) $\Rightarrow\ \alpha+\beta=\dfrac{\pi}{4} \Rightarrow \varepsilon\phi(\alpha+\beta)=\varepsilon\phi\dfrac{\pi}{4} \Rightarrow \dfrac{\varepsilon\phi\alpha+\varepsilon\phi\beta}{1-\varepsilon\phi\alpha\,\varepsilon\phi\beta}=1 \Rightarrow \varepsilon\phi\alpha+\varepsilon\phi\beta=1-\varepsilon\phi\alpha\,\varepsilon\phi\beta$

$\Rightarrow \dfrac{\pi}{x+1}+\dfrac{1}{1+2x}=1-\dfrac{\pi}{x+1}\cdot\dfrac{1}{1+2x} \Rightarrow \pi+2\pi x+\cancel{x}+\cancel{1}=\cancel{x}+2x^2+\cancel{1}+2x-\pi \Rightarrow \Rightarrow$

$(x-\pi)(x+1)=0 \Rightarrow x=-1$ απορ., $x=\pi$ δεκτή, διότι επαληθεύει την εξίσωση.

4. $\dfrac{dy}{dx}=e^{-y}\cdot\eta\mu\chi\cdot\eta\mu2\chi \Rightarrow \int e^y dy=\int\eta\mu\chi\cdot\eta\mu2\chi\,dx \Rightarrow e^y=\int\eta\mu\chi\cdot2\cdot\eta\mu\chi\cdot\sigma\upsilon\nu\chi\,dx \Rightarrow$

$e^y=2\int\eta\mu\chi^2\,d(\eta\mu\chi) \Rightarrow e^y=\dfrac{2}{3}\eta\mu^3\chi+c \Rightarrow y=\ln\left|\dfrac{2}{3}\eta\mu^3x+c\right|$

5. 1,1, 1,2.2,3.

(α) 1+1+3=5 $\Rightarrow P(a_1)=\dfrac{3}{6}\cdot\dfrac{3}{6}\cdot\dfrac{1}{6}\cdot\dfrac{3!}{2!}=\dfrac{1}{8}$, $\qquad$ 1+2+2=5 $\Rightarrow P(a_2)=\dfrac{3}{6}\cdot\dfrac{2}{6}\cdot\dfrac{2}{6}\cdot\dfrac{3!}{2!}=\dfrac{1}{6}$

$P(a)=P(a_1)+P(a_2)=\dfrac{1}{8}+\dfrac{1}{6}=\dfrac{7}{24}$

(β) $P(\beta)=P(\beta'_1)\cdot P(\beta'_2)\cdot P(\beta_3)=\dfrac{4}{6}\cdot\dfrac{4}{6}\cdot\dfrac{2}{6}=\dfrac{4}{27}$

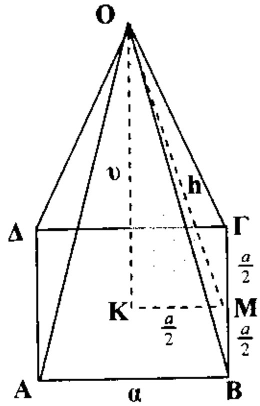
6. $A=\begin{pmatrix}3 & 6\\ -1 & -2\end{pmatrix}$ και $B=\begin{pmatrix}6 & 6\\ -1 & 1\end{pmatrix}$.

(α) $A^2 = A\cdot A = \begin{pmatrix} 3 & 6 \\ -1 & -2 \end{pmatrix}\cdot\begin{pmatrix} 3 & 6 \\ -1 & -2 \end{pmatrix} = \begin{pmatrix} 3 & 6 \\ -1 & -2 \end{pmatrix} = A$.

(β) $A^5 = \left(A^2\right)^2\cdot A = A^2\cdot A = A\cdot A = A^2 = A$

$A^5 + \lambda I = B \Rightarrow A + \lambda I = B \Rightarrow \begin{pmatrix} 3 & 6 \\ -1 & -2 \end{pmatrix} + \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 6 & 6 \\ -1 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 3+\lambda & 6 \\ -1 & -2+\lambda \end{pmatrix} = \begin{pmatrix} 6 & 6 \\ -1 & 1 \end{pmatrix}$

$\Rightarrow 3+\lambda = 6,\ \ -2+\lambda = 1 \quad \Rightarrow \lambda = 3$



7. $3\upsilon + 4\alpha = 24 \Rightarrow \upsilon = \dfrac{24-4\alpha}{3} \Rightarrow V_{(a)} = \dfrac{1}{3}a^2\cdot\dfrac{24-4a}{3} \Rightarrow V_{(a)} = \dfrac{4}{9}\left(6a^2 - a^3\right)$

$\Rightarrow \dfrac{dV}{da} = \dfrac{4}{9}\left(12a - 3a^3\right),\ \dfrac{dV}{da} = 0 \Leftrightarrow\ 12a - 3a^2 = 0 \Leftrightarrow a\left(4-a\right) = 0 \Rightarrow$

$\alpha = 0$ (απορ.) ή $\alpha = 4$. $\dfrac{d^2V}{da^2} = \dfrac{4}{9}\left(12 - 6a\right)$

$\left.\dfrac{d^2V}{da^2}\right|_{a=4} = \dfrac{4}{9}\left(12-24\right) < 0 \Rightarrow$ μέγιστη τιμή για α=4. $\Rightarrow \upsilon = \dfrac{8}{3}$.

8. (α) $\left.\begin{array}{l} y^2 = 4x \\ x^2 + y^2 + 2x - 7 = 0 \end{array}\right\} \Rightarrow x^2 + 4x + 2x - 7 = 0 \Rightarrow x^2 + 6x - 7 = 0 \Rightarrow \left(x-1\right)\left(x+7\right) = 0$

$\Rightarrow x = -7,\ x = 1$ (χ>0) $x = 1 \Rightarrow y = 2 \Rightarrow A\left(1,2\right)$

(β) $y^2 = 4x \Rightarrow 2y\dfrac{dy}{dx} = 4 \Rightarrow$

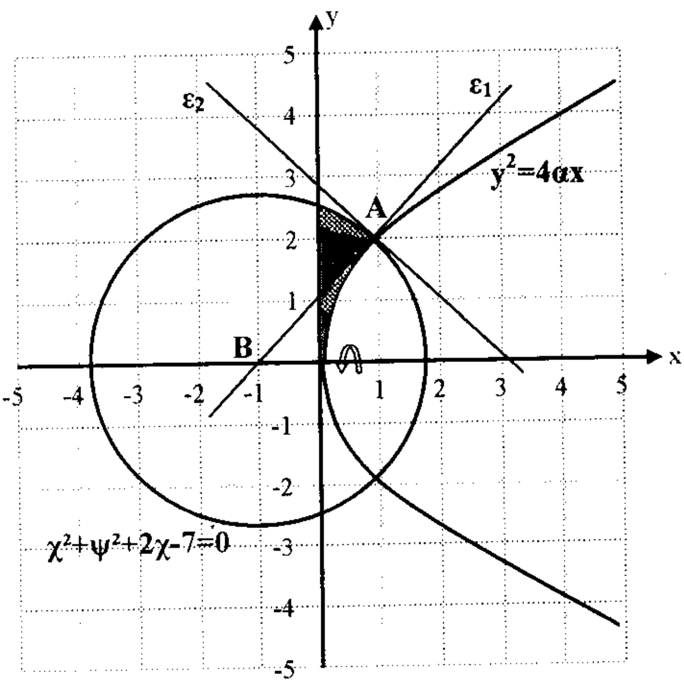
$\dfrac{dy}{dx} = \dfrac{2}{y} \Rightarrow \lambda_{\varepsilon_1} = \dfrac{2}{2} = 1$

$x^2 + y^2 + 2x - 7 = 0 \Rightarrow$

$2x + 2y\dfrac{dy}{dx} + 2 = 0 \Rightarrow \dfrac{dy}{dx} = \dfrac{-1-x}{y} \Rightarrow$

$\lambda_{\varepsilon_2} = \dfrac{-1-1}{2} = -1$

$\lambda_{\varepsilon_1}\cdot\lambda_{\varepsilon_2} = 1\cdot\left(-1\right) = -1 \Rightarrow$ οι καμπύλες τέμνονται ορθογώνια.

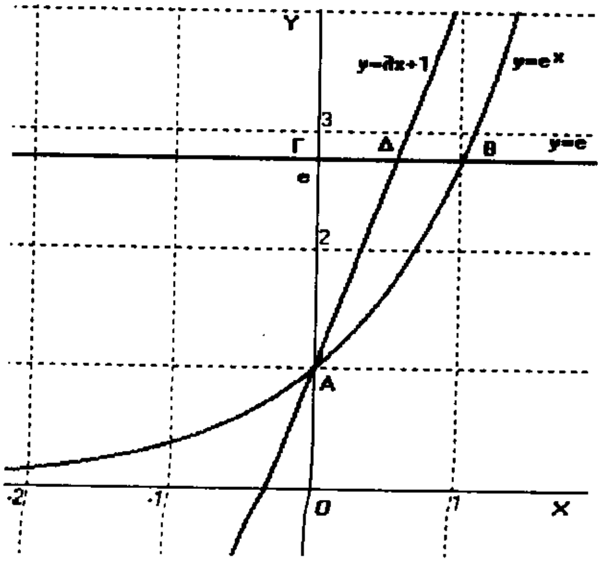


(γ) $V = V_1 - V_2 \Rightarrow$

$V = \pi\int_0^1\left(7 - 2\chi - \chi^2\right)dx - \pi\int_0^1 4\chi dx \Rightarrow$

$V = \pi\left[7x - 3x^2 - \dfrac{x^3}{3}\right]_0^1 \Rightarrow V = \dfrac{11\pi}{3}\ \kappa.\mu.$

9. $y=e^x,\ \ y=e \Rightarrow e=e^x \Rightarrow x=1 \Rightarrow B(1,e)$



$$E_{AB\Gamma}=\int_0^1\left(e-e^x\right)dx=\left[ex-e^x\right]_0^1=1 \text{ τ.μ.}$$

Σημείο Δ: $\left.\begin{array}{l} y=e \\ y=\lambda x+1 \end{array}\right\} \Rightarrow \lambda x+1=e \Rightarrow x=\dfrac{e-1}{\lambda}$

$$\Rightarrow \Delta\left(\frac{e-1}{\lambda},e\right)$$

$$E_{\overset{\triangle}{A\Gamma\Delta}}=\frac{1}{2}(A\Gamma)\cdot(\Gamma\Delta)=\frac{1}{2}(e-1)\frac{e-1}{\lambda}=\frac{(e-1)^2}{2\lambda}$$

$$E_{AB\Gamma}=2E_{A\Gamma\Delta} \Rightarrow 1=\cancel{2}\frac{(e-1)^2}{\cancel{2}\lambda} \Rightarrow \boxed{\lambda=(e-1)^2}$$

10. Έστω χ η τιμή του κόστους εισαγωγής $\Rightarrow \left(\dfrac{190}{100}x+1000\right)\cdot\dfrac{110}{100}=9460 \Rightarrow$

$(19x+10000)\cdot 11=946000 \Rightarrow 209x=836000 \Rightarrow x=4000$.

Η νέα τιμή για τον αγοραστή είναι: $\left(\dfrac{155}{100}\cdot 4000+1000\right)\cdot\dfrac{113}{100}=8136$. Η νέα τιμή είναι £8136.

ΜΕΡΟΣ Β'

1. $f(x)=\dfrac{x^2-1}{x^2-4}$ π.ο. $x\in\mathbb{R}-\{-2,2\}$

Τομές με άξονες: για χ=0 $\Rightarrow y=\dfrac{1}{4} \Rightarrow \left(0,\dfrac{1}{4}\right)$, Για y=0 $\Rightarrow x=\pm 1 \Rightarrow (-1,0)$, $(1,0)$

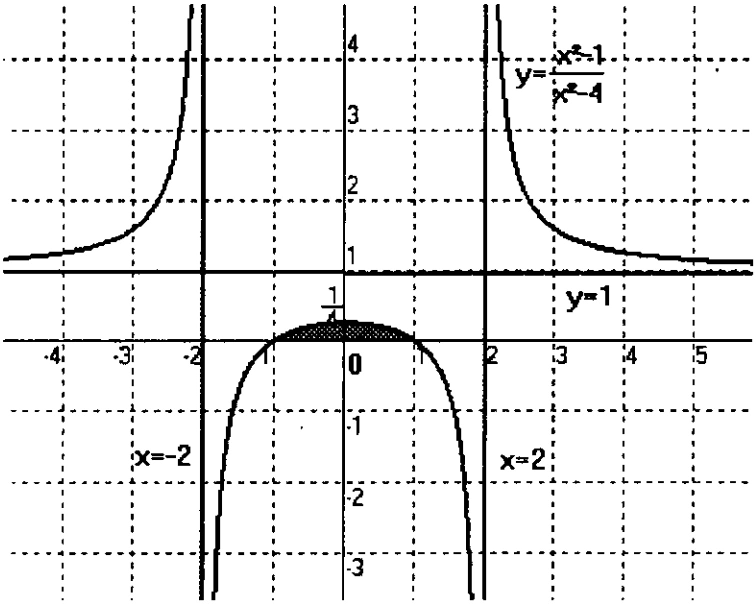
$\left.\begin{array}{l} \lim\limits_{x\to+\infty} f(x)=1 \\ \lim\limits_{x\to-\infty} f(x)=1 \end{array}\right\} \Rightarrow y=1$ οριζόντια ασύμπτωτη, $x^2-4=0 \Rightarrow x=\pm 2 \Rightarrow$ $\begin{array}{l} x=2 \\ x=-2 \end{array}$ κατακόρυφες ασύμπτωτες

$$f'(x)=\frac{2x\left(x^2-4\right)-2x\left(x^2-1\right)}{\left(x^2-4\right)^2}=\frac{-6x}{\left(x^2-4\right)^2}, \qquad f'(x)=0 \Rightarrow x=0$$

x	$-\infty$		-2		0		2		$+\infty$
f'(x)		+	‖	+	0	−	‖	−	
f(x)		↗	‖	↗	max	↘	‖	↘	

Για $\chi=0 \Rightarrow y_{\max}=\dfrac{0-1}{0-4}=\dfrac{1}{4}$

$\max\left(0,\dfrac{1}{4}\right)$



$$\frac{3}{x^2-4}=\frac{A}{x-2}-\frac{B}{x+2}\Rightarrow$$

$$A=\frac{3}{4}, B=-\frac{3}{4}\Rightarrow$$

$$\frac{3}{x^2-4}=\frac{3}{4(x-2)}-\frac{3}{4(x+2)}$$

$$E=2\int_0^1\left(1+\frac{3}{x^2-4}\right)dx\Rightarrow$$

$$E=2\int_0^1\left(1+\frac{3}{4(x-2)}-\frac{3}{4(x+2)}\right)dx$$

$$E=\left[2x+\frac{3}{2}\ln\left|\frac{x-2}{x+2}\right|\right]_0^1\Rightarrow$$

$$E=\left(2+\frac{3}{2}\ln\frac{1}{3}\right)-\left(0+\frac{3}{2}\ln 1\right)\Rightarrow E=\left(2-\frac{3}{2}\ln 3\right)\tau.\mu.$$

2. ΕΛΕΥΘΕΡΙΑ , Αναγραμματισμοί: $\frac{9!}{3!}=60480$

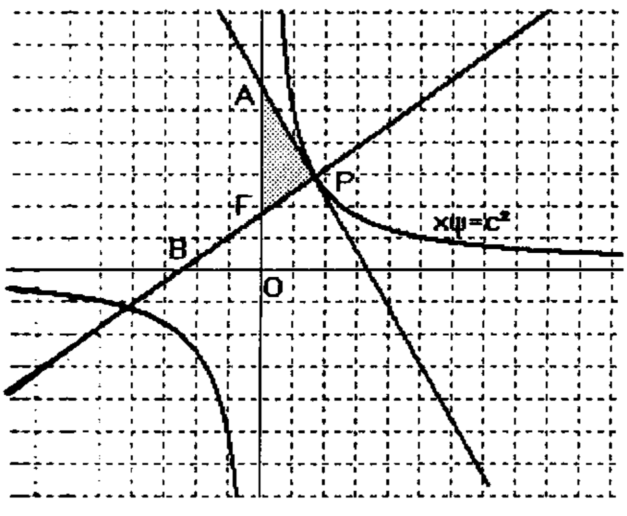
Ε	7!	Ε

Α $\Rightarrow P(A)=\frac{7!}{\frac{9!}{3!}}=\frac{3!\cdot 7!}{9!}\Rightarrow \boxed{P(A)=\frac{1}{12}}$

Β Δυνατές περιπτώσεις: 7!, ευνοϊκές περιπτώσεις: 6!

Ε . . . Ε . . . Ε $P(B)=\frac{6!}{7!}\Rightarrow\boxed{P(B)=\frac{1}{7}}$

3. (α) $xy=c^2\Rightarrow y+x\frac{dy}{dx}=0\Rightarrow\frac{dy}{dx}=-\frac{x}{y},\ P\left(cp,\frac{c}{p}\right)\Rightarrow \lambda_\varepsilon=-\frac{\frac{c}{p}}{cp}=-\frac{1}{p^2}$ και $\lambda_\kappa=p^2$



$$(\varepsilon): y-\frac{c}{p}=-\frac{1}{p^2}(x-cp)\Rightarrow x+p^2y=2cp$$

$$x=0\Rightarrow y=\frac{2c}{p}\Rightarrow A\left(0,\frac{2c}{p}\right)$$

$$(\kappa): y-\frac{c}{p}=p^2(x-cp)\Rightarrow$$

$$p^3x-py=c\left(p^4-1\right),$$

$$y=0\Rightarrow x=c\frac{p^4-1}{p^3}\Rightarrow B\left(c\frac{p^4-1}{p^3},0\right),\qquad x=0\Rightarrow y=-c\frac{p^4-1}{p}\Rightarrow \Gamma\left(0,-c\frac{p^4-1}{p}\right)$$

$$\left.\begin{aligned} x_M&=\frac{x_A+x_B}{2}=\frac{0+c\frac{p^4-1}{p^3}}{2}=\frac{c\left(p^4-1\right)}{2p^3}\\ y_M&=\frac{y_A+y_B}{2}=\frac{\frac{2c}{p}+0}{2}=\frac{c}{p}\Rightarrow p=\frac{c}{y}\end{aligned}\right\}\Rightarrow x=\frac{x\left(\left(\frac{c}{p}\right)^4-1\right)}{2\left(\frac{c}{p}\right)^3}\Rightarrow \Gamma.\mathrm{T}.\ \boxed{2c^2xy=c^4-y^4}$$

(β)
$$\left.\begin{aligned} E_{\overset{\Delta}{(\mathrm{P}A\Gamma)}}&=\frac{1}{2}\left|x_p\right|\cdot\left|y_A-y_\Gamma\right|\\ E_{\overset{\Delta}{(\mathrm{P}A\Gamma)}}&=\frac{17c^2}{2}\end{aligned}\right\}\Rightarrow \frac{1}{2}cp\left(\frac{2c}{p}+c\frac{p^4-1}{p}\right)=\frac{17c^2}{2}\Rightarrow p^4=1\Rightarrow \begin{matrix}p=\pm 2\\ p>0\end{matrix}\Rightarrow \boxed{p=2}$$

4. $u=e^y\Rightarrow \frac{du}{dx}=e^y\cdot\frac{dy}{dx}\Rightarrow \frac{dy}{dx}=\frac{1}{e^y}\cdot\frac{du}{dx}\Rightarrow \frac{dy}{dx}=\frac{1}{u}\cdot\frac{du}{dx}\qquad x\frac{dy}{dx}+x+1=xe^{x-y}\Rightarrow$

$$x\left(\frac{1}{u}\cdot\frac{du}{dx}\right)+x+1=x\frac{e^x}{e^y}\Rightarrow \frac{x}{u}\cdot\frac{du}{dx}+x+1=x\frac{e^x}{u}\Rightarrow x\frac{du}{dx}+\left(x+1\right)u=xe^x\Rightarrow \frac{du}{dx}+\frac{x+1}{x}u=e^x,$$

$$I\left(x\right)=e^{\int P(x)dx}\Rightarrow I\left(x\right)=e^{\int\frac{x+1}{x}dx}\Rightarrow I\left(x\right)=e^{\int\left(1+\frac{1}{x}\right)dx}\Rightarrow I\left(x\right)=e^{x+\ln x}\Rightarrow I\left(x\right)=e^x\cdot e^{\ln x}\Rightarrow \underline{I\left(x\right)=xe^x}$$

$$xe^x\frac{du}{dx}+xe^x\frac{x+1}{x}u=xe^xe^x\Rightarrow \frac{d}{dx}\left(xe^xu\right)=xe^{2x}\Rightarrow \int d\left(xe^xu\right)=\int xe^{2x}dx\Rightarrow$$

$$xe^xu=\int xd\left(\frac{1}{2}e^{2x}\right)\Rightarrow xe^xu=\frac{1}{2}xe^{2x}-\frac{1}{2}\int e^{2x}dx\Rightarrow xe^xu=\frac{1}{2}xe^{2x}-\frac{1}{4}e^{2x}+c\Rightarrow$$

$$u=\frac{1}{2}e^x-\frac{1}{4}\frac{e^x}{x}+\frac{c}{x}e^{-x}\Rightarrow e^y=\frac{1}{2}e^x-\frac{e^x}{4x}+\frac{c}{x}e^{-x}\Rightarrow y=\ln\left|\frac{1}{2}e^x-\frac{e^x}{4x}+\frac{c}{x}e^{-x}\right|$$

5. $I\left(a,\beta\right)=\int_\alpha^\beta\frac{1-\chi^2}{\left(1+\chi^2\right)\sqrt{1+\chi^4}}dx\ ,\quad x=\frac{1}{u}\Rightarrow dx=-\frac{1}{u^2}du$

x	a	β
u	$\frac{1}{a}$	$\frac{1}{\beta}$

$$I\left(a,\beta\right)=\int_{\frac{1}{a}}^{\frac{1}{\beta}}\frac{1-\frac{1}{u^2}}{\left(1+\frac{1}{u^2}\right)\sqrt{1+\frac{1}{u^4}}}\left(-\frac{1}{u^2}\right)du=-\int_{\frac{1}{a}}^{\frac{1}{\beta}}\frac{u^2-1}{\left(u^2+1\right)\frac{\sqrt{u^4+1}}{\cancel{u^2}}}\cdot\frac{1}{\cancel{u^2}}du=\int_{\frac{1}{a}}^{\frac{1}{\beta}}\frac{1-u^2}{\left(1+u^2\right)\sqrt{u^4+1}}du$$

$$=I\left(\frac{1}{a},\frac{1}{\beta}\right)\ \text{άρα}\ I\left(a,\beta\right)=I\left(\frac{1}{a},\frac{1}{\beta}\right)\quad(1)$$

$$I\left(\frac{1}{a},a\right)\overset{(1)}{=}I\left(\frac{1}{\frac{1}{a}},\frac{1}{a}\right)=I\left(a,\frac{1}{a}\right),\qquad I\left(a,\beta\right)=-I\left(\beta,a\right)\Rightarrow I\left(a,\frac{1}{a}\right)=-I\left(\frac{1}{a},a\right)$$

$$\left.\begin{aligned} I\left(\frac{1}{a},a\right)&=I\left(a,\frac{1}{a}\right)\\ I\left(a,\frac{1}{a}\right)&=-I\left(\frac{1}{a},a\right)\end{aligned}\right\}\Rightarrow I\left(\frac{1}{a},a\right)=-I\left(\frac{1}{a},a\right)\Rightarrow 2I\left(\frac{1}{a},a\right)=0\Rightarrow \boxed{I\left(\frac{1}{a},a\right)=0}$$